Graphs without even holes or diamonds

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Abstract. An even hole is an induced chordless cycle of even length at least four. A diamond is an induced subgraph isomorphic to $K_4 - e$. We show that graphs without even holes and without diamonds can be decomposed via clique-separators into graphs that have uniformly bounded cliquewidth.

1 Introduction

We consider undirected graphs without loops or multiple edges.

Definition 1. *A* chord *in a cycle is an edge connecting two vertices of the cycle that are not adjacent in the cycle. A cycle is* chordless *if it has no chord. A* hole *is a chordless cycle of length at least four. A graph is* chordal *if it has no holes.*

Similarly, a *chordless path* in a graph G is a set of vertices P such that $G[P]$ is a path. A hole is *even* if it has even length. A *diamond* is an induced $K_4 - e$. In this paper we consider graphs without even holes and diamonds.

Graphs without even holes were studied in [1,8,9,10,11,14,15]. These graphs can be recognized in polynomial time [8,11]. It was shown that every graph without even holes has a *bisimplicial extreme*, that is a vertex whose neighborhood is the union of two cliques [9]. Graphs without diamonds nor even holes were first studied in [21]. We showed in [21] that every graph without diamonds and without even holes has a *simplicial extreme*, that is a vertex which is either simplicial or which has degree 2.

Cliquewidth was introduced by Courcelle and Olariu in [12]. For integer k we define the following k-*composers*:

1. For $i \in \{1, \ldots, k\}$, $i(x)$ creates a vertex x with label i.
2. For distinct $i, j \in \{1, \ldots, k\}$, let $\eta_{i,j}$ be the operation which adds all edges joining vertices with label i to vertices with label j.
3. For $i, j \in \{1, \ldots, k\}$, let $\rho_{i \to j}$ be the operation which relabels every vertex with label i with the label j.
4. The operation $G \oplus H$ creates the graph which is the disjoint union of two labeled graphs G and H.

Definition 2. *A graph* G *has* cliquewidth k *if* G *can be constructed via a series of* k*-composers.*

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The corresponding graph-decomposition is called a k-expression. Courcelle showed that those problems that can be formulated in monadic second-order logic without quantification over subsets of edges, can be solved efficiently for graphs of bounded cliquewidth (see [13]). For these algorithms a k-expression is a necessary ingredient. However, this obstacle was taken away with the introduction of rankwidth. The graph classes for which the parameters cliquewidth and rankwidth are bounded coincide: It was shown in [25,27] that $rw \leqslant cw \leqslant 2^{rw+1} - 1$. An algorithm that computes a rank-decomposition-tree for a graph of bounded rankwidth was obtained in [18,19]. This algorithm runs in $O(n^3)$ time. Interestingly, it is still unknown whether CLIQUEWIDTH is fixed-parameter tractable (see, *e.g.*, [17]). Computing cliquewidth is NP-complete [16]. The NP-completeness of RANKWIDTH seems to follow from arguments given in [19].

In this paper we show that graphs without even holes and without diamonds can be decomposed via clique separators into graphs which have a uniform bound on the cliquewidth. It easily follows that graphs in this class can be recognized in $O(n^3)$ time. This result was anticipated by [15]. That paper shows that planar graphs without even holes have uniformly bounded treewidth.

2 Birdcages and links

For two sets A and B we write $A + B$ for $A \cup B$ and $A - B$ for $A \setminus B$. For a set A and an element x we also write $A + x$ instead of $A + \{x\}$. For a vertex x we write $N(x)$ for the set of its neighbors and we write $N[x] = x + N(x)$ for its *closed* neighborhood. For a subset S of vertices we write $N(S) = \bigcup_{x \in S} N(x) - S$ and we write $N[S] = N(S) + S$. For a subset S of vertices of a graph G we write $G[S]$ for the subgraph of G *induced* by S. For a graph $G = (V, E)$ and a subset S of its vertices we write $G - S$ for the graph $G[V - S]$. If S consists of a single vertex x we also write $G - x$ instead of $G - \{x\}$.

Let $\mathcal{G}$ be the class of graphs without even holes and without diamonds.

Definition 3. *A* simplicial *is a vertex whose neighborhood induces a clique. A* simplicial extreme *is a vertex which either is simplicial, or has degree two.*

Lemma 1 ([21]). *Every graph which has no even holes nor diamonds is either a clique or has two simplicial extremes that are not adjacent.*

Definition 4. *Let C_1 and C_2 be cliques. Consider all chordless paths from a vertex of C_1 to a vertex of C_2 which does not use any other vertex of C_1 or C_2. If the lengths of all those paths have the same parity, then this collection of chordless paths is called a* C_1, C_2-link*.*

In this section we analyze the structure of links. Our basic building blocks are birdcages.

Definition 5. *A* birdcage *is a graph with a specified clique F, called the* floor*, and a specified vertex $h \notin F$, called the* hook*, and a collection of paths $\mathcal{P}$, one from each vertex in F to h, such that*

(a) *all path of $\mathcal{P}$ have odd length or all paths in $\mathcal{P}$ have even length, and*
(b) *either at most one path in $\mathcal{P}$ has length one or all paths in $\mathcal{P}$ have length one.*

We consider two ways to connect two birdcages.

Definition 6. *Let $B_i = (F_i, h_i)$, $i = 1, 2$ be two birdcages. A* join *of B_1 and B_2 is the graph obtained either by adding all edges between the floor or the hook of B_1 and the floor or the hook of B_2, or by identifying the two hooks h_1 and h_2. In case $|F_1| = |F_2|$ and if exactly one of B_1 and B_2 is a clique, then we also call the vertex-by-vertex identification of F_1 and F_2 a join. In case $|F_1| = |F_2| = 2$ and if for $i = 1, 2$ there exists exactly one vertex $x_i \in F_i$ which is adjacent to h_i, then we also call the identification of F_1 and F_2 such that x_1 and x_2 are identified, a join.*

Definition 7. *Let $B = (F, h)$ be a birdcage. Let h^* be a fixed neighbor of h such that h and h^* have no neighbors in common. The edge (h, h^*) is called the* skew-edge *of B.*

Definition 8. *Let $B_i = (F_i, h_i)$, $i = 1, 2$ be two birdcages each with a skew-edge (h_i, h_i^*). A* skew-join *of B_1 and B_2 is the graph obtained by identifying each of h_i and h_i^* with one of h_{3-i} and h_{3-i}^*. Let $B_1 = (F_1, h_1)$ be a birdcage and let $B_2 = (F_2, h_2)$ be a birdcage with a skew-edge (h_2, h_2^*). The identification of h_1 with one of h_2, h_2^*, and also the join of h_1 or F_1 with one of h_2 and h_2^*, is called a skew-join.*

We consider one more operation.

Definition 9. *Let $B = (F, h)$ be a birdcage and let L be a link from a vertex to a clique. Assume that the length of L has the same parity as the lengths of the paths connecting h with F in B. A* replacement *is the operation of substituting one h, F-path in B by L. The clique of L is joined to the other vertices of F.*

We call a birdcage in which some paths have been replaced by links again a birdcage.

Theorem 1. *There exists a natural number t such that every link has cliquewidth at most t.*

Proof. Consider a link L between two nonadjacent vertices x and y. Assume first that x has at least two nonadjacent neighbors. Between any pair of cliques C_1 and C_2 in $N(x)$ all chordless paths must be odd thus the induced C_1, C_2-paths in L form a C_1, C_2-link. Note that y is not in any of these induced sublinks since that implies an even hole. Let L^* be the subgraph of L induced by these sublinks. We prove that L^* induces a birdcage with hook x and some separating floor F. The floor F separates L^* from a link from F to y. Consider the component C of $L - L^*$ that contains y. We first show that $N(C)$ is a clique. Let a and b be nonadjacent vertices in $N(C)$. Since an a, b-path with internal vertices in C cannot connect two vertices that are in a link, a and b must be two nonadjacent vertices in a birdcage. However, by induction this implies that there is a link (either between

two cliques in $N(x)$ or from a clique in $N(x)$ to y) which contains a birdcage with two ends of the same type $\in \{\text{hook}, \text{floor}\}$, which is a contradiction. The chordless paths from y to any of the cliques in $N(x)$ induce a link. By induction, either there exists a join between two cliques $N(C)$ and $N(L^*)$ and two links between $N(C)$ and x and $N(L^*)$ and y or there exists a skew-edge (a, b). The skew-edge connects three links; two to $N(x)$ and one to y. Note however that such a skew-join is only possible in case $N(x)$ is a clique (see below) since otherwise there would exist an even hole.

In case $N(x)$ is a clique, we obtain a similar decomposition, except that in this case y can be a vertex of L^*, which in this case is defined as follows. When $N(x)$ is a clique we consider the subgraph L^* induced by chordless paths between different vertices of $N(x)$, with internal vertices in $L - N[x]$. The structure is proved by induction on $|N(x)|$. Note that a skew-connection between L^* and $L - L^*$ is only possible in case $N(x)$ is a clique.

The induced subgraph L^* is a *bag*. It follows that L can be decomposed into a tree of bags glued together along clique cutsets either via joins or via skew-joins.

Consider the *cross-edges* in an x, y-link L for nonadjacent vertices x and y: Assume two vertices a and b are adjacent but the edge is not an edge of L. Then a and b are contained in a birdcage, since one cannot be on a chordless x, y-path of the other. It follows that there exists a birdcage $B = (F, h)$ with a vertex $a \in F$ adjacent to h, and the vertex a is adjacent to some cliques in the other (replaced) h, F-paths. By induction on the induced link-structures, the chordless paths from a to either x or y must form a link. This implies that if a is adjacent to two vertices in different paths of a birdcage $B' = (F', h')$ then a is also adjacent to h'. Note also that for each birdcage B^* there is at most one vertex a (adjacent to the hook of B^*) which has crossing edges to paths of B^*. This proves that these birdcages with their cross-edges can be described by a bounded cliquewidth expression.

By the recursive definition, this proves the theorem. □

3 Link-extensions

In this section we extend links in a recursive way. Consider two nonadjacent vertices a and b in an x, y-link L induced by the collection of chordless x, y-paths in a graph $G \in \mathcal{G}$. Assume there exists a chordless a, b-path with internal vertices $\notin L$. Then a and b must be vertices of a birdcage, since they cannot both lie on a common chordless x, y-path. Thus there exists a birdcage $B = (F, h)$ with a vertex $a \in F$ which is adjacent to h. There are (extended) links, "cross-links," from a to cliques or skew-joins in the other replaced paths in B. These cross-links act as the cross-edges introduced in the proof of Theorem 1, except that the cross-links can also go to skew-edges. Note that if a has links to cliques or skew-joins in different paths of some birdcage B' then a must also be adjacent to the hook of B'. Furthermore, for each constituent birdcage B^* in L there is at most one vertex a with links to paths of B^*.

Theorem 2. *There exists a number* t *such that every graph* $G \in \mathcal{G}$ *without clique-separators has cliquewidth* t.

Proof. Assume G is not a clique. We proved in [21] that G has a vertex ω with two nonadjacent neighbors x and y. The collection of chordless x, y-paths forms an x, y-link in $G - \omega$. Let L^* be the extension of L in $G - \omega$. By definition of a link-extension, and since there are no clique-separators, $G - \omega = L^*$. □

4 Cliquewidth of (even-hole,diamond)-free graphs

Definition 10. *A* splitgraph *is a graph* $H = H(C, I, E)$ *with a partition of the vertices in a clique* C *and an independent set* I.

Lemma 2 ([3,22]). *Cliquewidth is unbounded for splitgraphs.*

Definition 11. *A* birdcage-split *is a graph which can be constructed from a splitgraph* $H = H(C, I, E)$ *by replacing all edges incident with every vertex* $x \in I$ *either by an even or an odd* > 1*-length path.*

Thus a birdcage-split H consists of a clique C and a collection $\mathcal{F}$ of subsets of C and an independent set $\mathcal{H}$ of hooks. Each hook forms a birdcage in H with floor $F \in \mathcal{F}$.

Let $\mathcal{BS}$ be the class of birdcage-splits.

Lemma 3. *Birdcage-splits do not have even holes or diamonds.*

Proof. Every hole is contained in a birdcage, thus it is odd. Since all paths in birdcages have length more than one, there is no diamond. □

Lemma 4. *For every natural number* t *there exists a graph in* $\mathcal{BS}$ *with cliquewidth more than* t.

Proof. Let $G = G(C, I, E)$ be a splitgraph and let G^* be the birdcage-split obtained from G by subdividing every edge (x, y) with $x \in C$ and $y \in I$ by a single vertex. Let z be a subdivision vertex of G^* and consider a *local complementation* at z. If $x \in C$ and $y \in I$ are the two neighbors of z then this adds the edge (x, y) to G^*. Let $\hat{G}$ be the graph obtained from G^* by doing a local complementation at every subdivision vertex. Then G is an induced subgraph of $\hat{G}$, that is, G is a *vertex-minor* of G^*. This implies that the rankwidth of G^* is at least the rankwidth of G [26]. By Lemma 2 this proves the claim. □

Theorem 3. *There exists a natural number* t *such that every* $G \in \mathcal{G}$ *either has a clique-separator or has cliquewidth at most* t.

Proof. Links and link-extensions can be generated by a recursive function of bounded width. The graph can be decomposed by clique-separators into these link-extensions. □

Recall that a decomposition by clique separators can be obtained in $O(n^3)$ time [34].

Corollary 1. *Graphs in* $\mathcal{G}$ *can be recognized in* $O(n^3)$ *time.*

5 Remarks on geodetic graphs

Geodetic graphs were introduced by Ore.

Definition 12 ([4,24]). *A graph is* geodetic *if for every pair of vertices the shortest path between them is unique.*

Note that a graph is geodetic if and only if for every vertex x, every vertex $y \in N_k(x)$[1] is adjacent to exactly one vertex in $N_{k-1}(x)$, $k \geqslant 2$; see [28]. This settles the recognition problem. It follows from the definition that geodetic graphs have no induced diamond and no induced C_4.

One partial characterization of geodetic graphs of diameter two appeared in [32]. See also [2,33,36]. These contain the Moore graphs of diameter two, *i.e.*, the 5-cycle, the Petersen graph and the Hoffman-Singleton graph.[2] Note that the Petersen graph has an induced C_6. It can be shown (see [20,32,30,5] and [6, Theorem 1.17.1]) that, if G is a geodetic graph of diameter two then either

(i) G contains a universal vertex,[3] or
(ii) G is strongly regular, or
(iii) G has exactly two vertex degrees $k_1 > k_2$. Let X_i be the set of vertices with degree k_i. Then X_2 is an independent set. Every maximal clique that contains a vertex of X_1 and X_2 has size two. Every maximal clique contained in X_1 has size $k_1 - k_2 + 2$. Furthermore, $n = k_1 k_2 + 1$.

Plesník [29] and Stemple [33] show that a geodetic graph G is homemorphic to K_n if and only if there exists a function f which assigns a nonnegative integer to every vertex of K_n such that an edge (x, y) in K_n has $f(x) + f(y)$ extra vertices in G. See [7, Section 7.3] for various constructions of geodetic graphs. A characterization of planar geodetic graphs appeared in [31,35].

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[2] The only other possible degree that a Moore graph can have (thus with diameter 2 and girth 5) is 57. The existence of such a graph is unsettled [23]. The smallest unsettled case for a strongly regular graph with $\mu = 1$ has parameters $(n, k, \lambda, \mu) = (400, 21, 2, 1)$ [6].
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